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Inference for Incomplete Paired Functional Data: Two Approaches

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Powerful inference for functional data in complex factorial designs

OUTCLASS Research Group



Statistical Question: Two-sided Missingness in Functional Two-sample Testing

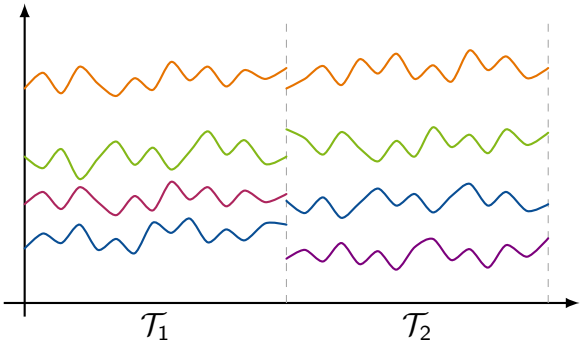
Motivation: Missing values in repeated measurements are unpleasant

- Examples: non-responding in surveys at one time point, lost patients in clinical studies.

$\longrightarrow$ With naive handling, all measured time points must be omitted.

$\longrightarrow$ Many options to handle missing values, mostly based on imputation.

- Handling of missingness in functional data remains challenging.



Missingness in Functional Data

Functional data in a matched pairs design:
Suppose we have a pseudometric compact space $(\mathcal{T}, \rho)$ and $n \in \mathbb{N}$ independent realisations

$$\mathbf{x}_1, \ldots, \mathbf{x}_n : \mathcal{T} \to \mathbb{R}^2$$

of a stochastic process $\mathbf{X}$ and write for every $t \in \mathcal{T}$:

$$\begin{pmatrix} x_{11}(t) \\ x_{21}(t) \end{pmatrix}, \ldots\ldots\ldots, \begin{pmatrix} x_{1n}(t) \\ x_{2n}(t) \end{pmatrix}.$$

Missingness in Functional Data

Functional data in a matched pairs design with missing functions:
Suppose we have a pseudometric compact space $(\mathcal{T}, \rho)$ and $n \in \mathbb{N}$ independent realisations

$$\mathbf{x}_1, \dots, \mathbf{x}_n : \mathcal{T} \to \mathbb{R}^2$$

of a stochastic process $\mathbf{X}$ and write for every $t \in \mathcal{T}$ and $n = n_1 + n_2 + n_3$:

$$\underbrace{\begin{pmatrix} x_{11}^{(c)}(t) \\ x_{21}^{(c)}(t) \end{pmatrix}, \dots, \begin{pmatrix} x_{1n_1}^{(c)}(t) \\ x_{2n_1}^{(c)}(t) \end{pmatrix}}_{\mathbf{x}^{(c)}, \text{ of size } n_1.}, \underbrace{\begin{pmatrix} x_{11}^{(m)}(t) \\ - - - \end{pmatrix}, \dots, \begin{pmatrix} x_{1n_2}^{(m)}(t) \\ - - - \end{pmatrix}, \begin{pmatrix} - - - \\ x_{21}^{(m)}(t) \end{pmatrix}, \dots, \begin{pmatrix} - - - \\ x_{2n_3}^{(m)}(t) \end{pmatrix}}_{\mathbf{x}^{(m)}, \text{ of size } n_2 + n_3.}.$$

Hypothesis

- Functional matched pair hypothesis regarding the pointwise means $\eta_1(t)$ and $\eta_2(t)$, $t \in \mathcal{T}$:

$$\mathcal{H}_0 : \forall\, t \in \mathcal{T} :\, \eta_1(t) = \eta_2(t), \quad \mathcal{H}_1 : \exists\, t \in \mathcal{T} :\, \eta_1(t) \neq \eta_2(t). \tag{1}$$

  $\longrightarrow$ Existing methods without missing values: Martínez-Camblor and Corral (2011) and Smaga (2020).

- Non-missing values in $\mathbf{x}^{(m)}$ are i.i.d.,
  $\longrightarrow$ missing observations are missing completely at random (MCAR).
  $\longrightarrow$ Classical imputation methods are not feasible.
- Aim: adapt the approach Amro and Pauly (2017) for the modelling of missing data in matched pairs for functional data.

Statistical Methods: Two-sided Missingness in functional Two-sample Testing

General Set-Up

- Consider stochastic process $\mathbf{X}$ is valued in $C^2(\mathcal{T})$:

$$\{\mathbf{X}(t) = (X_1(t), X_2(t)) : t \in \mathcal{T}\},\ \mathbf{X} : \Omega \to C^2(\mathcal{T}),\ \omega \mapsto \mathbf{X}_t(\omega).$$

  $\longrightarrow$ Similar to the approaches in Dette et al. (2020) and Munko et al. (2025).

- Assume no specific distribution, but mean function $\boldsymbol{\eta}(t) = (\eta_1(t), \eta_2(t))$, $t \in \mathcal{T}$, and covariance matrix

$$\mathbf{\Gamma}(s,t) = \begin{pmatrix} \gamma_1(s,t) & \gamma_{12}(s,t) \\ \gamma_{12}(s,t) & \gamma_2(s,t) \end{pmatrix}, s, t \in \mathcal{T}.$$

Assumptions

1. The covariance matrix $\mathbf{\Gamma}(t,t)$ is positive definite for all $t \in \mathcal{T}$.
2. The complete and incomplete parts do not vanish, i.e. $n_i/(n_2+n_3) \to \kappa_i > 0$ for $\min\{n_1,n_2,n_3\} \to \infty$, $i \in \{1,2,3\}$.

... Several technical assumptions on the subject-effect functions

$$\mathbf{v}_j(t) = \mathbf{x}_j(t) - \boldsymbol{\eta}(t), j \in \{1,\dots,n\},\ t \in \mathcal{T}.$$

$\longrightarrow$ Assumed to be i.i.d. realisations of a stochastic process $\{\mathbf{V}(t) : t \in \mathcal{T}\}$, $\mathbf{V} : \Omega \to \mathcal{C}^2(\mathcal{T})$ with zero mean function and covariance function $\mathbf{\Gamma}$.

$\longrightarrow$ Allow to use empirical process theory by van der Vaart and Wellner (2023).

Estimators: Pointwise Mean and Covariance

- Unbiased estimators for the mean function $\boldsymbol{\eta}$ is the point-wise mean:

$$\hat{\boldsymbol{\eta}}(t) = (\hat{\eta}_1(t), \hat{\eta}_2(t)) := \frac{1}{n}\sum_{j=1}^{n} \mathbf{x}_j(t),\ t \in \mathcal{T}.$$

- Empirical covariance function $\hat{\boldsymbol{\Gamma}} : \mathcal{T}^2 \to \mathbb{R}^{2\times 2}$ at point (s,t):

$$\begin{aligned}\hat{\boldsymbol{\Gamma}}(s,t) &= \begin{pmatrix} \hat{\gamma}_1(s,t) & \hat{\gamma}_{12}(s,t) \\ \hat{\gamma}_{12}(s,t) & \hat{\gamma}_2(s,t) \end{pmatrix} \\ &:= \frac{1}{n-1}\sum_{j=1}^{n} \left(\mathbf{x}_j(s) - \hat{\boldsymbol{\eta}}(s)\right)\left(\mathbf{x}_j(t) - \hat{\boldsymbol{\eta}}(t)\right)',\ s,t \in \mathcal{T}.\end{aligned}$$

$\longrightarrow$ Point-wisely consistent for $\Gamma(s,t)$ by the weak law of large numbers for every $s,t \in \mathcal{T}$.
$\longrightarrow$ Uniformly consistent over $\{(t,t) \in \mathcal{T}^2 \mid t \in \mathcal{T}\}$ under the assumptions by Munko et al. (2025).

The Test's Construction: Two Point-wise Test Statistics

For the complete data part $\mathbf{x}^{(c)}$, consider a paired two-sample test statistic:

$$T_{n_1}\left(\mathbf{x}^{(c)}, \hat{\mathbf{\Gamma}}^{(c)}\right)(t) := \sqrt{n_1}\,\frac{\hat{\eta}_1^{(c)}(t) - \hat{\eta}_2^{(c)}(t)}{\sqrt{\hat{\gamma}_1^{(c)}(t,t) - 2\hat{\gamma}_{12}^{(c)}(t,t) + \hat{\gamma}_2^{(c)}(t,t)}},$$

and for the missing data part $\mathbf{x}^{(m)}$, as suggested in Amro and Pauly (2017), consider a point-wise Welch-type test statistic:

$$T_{n_2+n_3}\left(\mathbf{x}^{(m)}, \hat{\mathbf{\Gamma}}_0^{(m)}\right)(t) := \frac{\hat{\eta}_1^{(m)}(t) - \hat{\eta}_2^{(m)}(t)}{\sqrt{\frac{\hat{\gamma}_1^{(m)}(t,t)}{n_2} + \frac{\hat{\gamma}_2^{(m)}(t,t)}{n_3}}}, t \in \mathcal{T}.$$

The Test's Construction: Combination I

Weight the point-wise squared test statistics for every $t \in \mathcal{T}$:

$$K_n(\mathbf{x}, t) = \sqrt{a_n}\, T^2_{n_1}\left(\mathbf{x}^{(c)}, \hat{\mathbf{\Gamma}}^{(c)}\right)(t) + \sqrt{1-a_n}\, T^2_{n_2+n_3}\left(\mathbf{x}^{(m)}, \hat{\mathbf{\Gamma}}_0^{(m)}\right)(t),$$

where $a_n \in [0,1]$, $a_n \to a$.
Global test statistics:

$$I_n(\mathbf{x}) = \int_{\mathcal{T}} K_n(\mathbf{x}, t)dt,$$
$$S_n(\mathbf{x}) = \sup_{t\in\mathcal{T}} K_n(\mathbf{x}, t).$$

The Test's Construction: Combination II

Theorem

Consider the assumptions.

1. *Under* $\mathcal{H}_0$*, the test statistics* $I(\mathbf{x})$ *and* $S(\mathbf{x})$ *converge in distribution:*

$$I_n(\mathbf{x}) \xrightarrow{d} Z^I = \int_{\mathcal{T}} \sqrt{a} Z_1^2 + \sqrt{1-a} Z_2^2 dt, \quad S_n(\mathbf{x}) \xrightarrow{d} Z^S = \sup_{t \in \mathcal{T}} \sqrt{a} Z_1^2 + \sqrt{1-a} Z_2^2, \quad (2)$$

   where $Z_1 \sim \mathrm{GP}(0, \mathrm{Cov}(Z_1))$ *and* $Z_2 \sim \mathrm{GP}(0, \mathrm{Cov}(Z_2))$ *with and* $Var(Z_i) = 1$*,* $i \in \{1, 2\}$*, for all* $s, t \in \mathcal{T}$*.*

2. *Under* $\mathcal{H}_1$*,* $I_n(\mathbf{x})$ *and* $S_n(\mathbf{x})$ *converge in probability to* ∞*.*

The Test's Construction: Testing Procedures

$\longrightarrow$ Resampling Methods need to be considered.

- **Permutation Procedure**: uniform limiting distribution of the permutation test statistic $(T^{2,*}_{n_2+n_3}(t))_{t\in\mathcal{T}}$ differs from the limiting distribution of $(T^2_{n_2+n_3}(t))_{t\in\mathcal{T}}$.
  $\longrightarrow$ Equality holds, if $n_2 = n_3$ or $\gamma_1(s,t) = \gamma_2(s,t)$ for all $s,t \in \mathcal{T}$.
- **Parametric Bootstrap Procedure** is theoretically valid by Janssen and Pauls (2003) and Munko et al. (2025).
- Testing Procedures for some resampling method $\circ$:

$$\varphi_n^{\circ,I} = \mathbf{1}\left\{I_n(\mathbf{x}) > q_{1-\alpha}^{\circ,I}(\mathbf{x})\right\}; \quad \varphi_n^{\circ,S} = \mathbf{1}\left\{S_n(\mathbf{x}) > q_{1-\alpha}^{\circ,S}(\mathbf{x})\right\}.$$

Statistical Question: One-sided Missingness in Functional Two-sample Testing

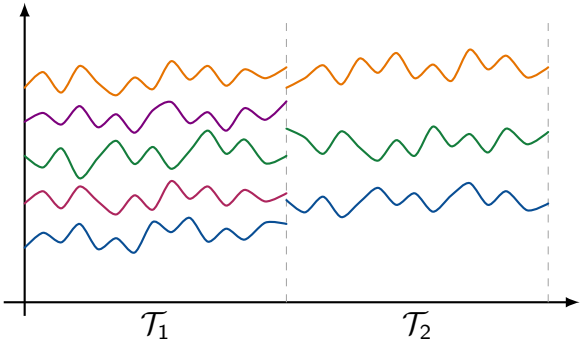
What if: there are only missing functions in one group

- Example: clinical study with functional measurement at two time points.

$\longrightarrow$ All patients who were treated at the first time point are included in the study.

$\longrightarrow$ Some do not appear at the second time point.

$\longrightarrow$ Goal: Find a test statistics that allows to consider also the unpaired data.



Missingness in Functional Data

Functional data in a matched pairs design:
Suppose we have a pseudometric compact space $(\mathcal{T}, \rho)$ and $n \in \mathbb{N}$ independent realisations

$$\mathbf{x}_1, \ldots, \mathbf{x}_n : \mathcal{T} \to \mathbb{R}^2$$

of a stochastic process $\mathbf{X}$ and write for every $t \in \mathcal{T}$:

$$\begin{pmatrix} x_{11}(t) \\ x_{21}(t) \end{pmatrix}, \ldots\ldots\ldots, \begin{pmatrix} x_{1n}(t) \\ x_{2n}(t) \end{pmatrix}.$$

Missingness in Functional Data

Functional data in a matched pairs design with missing functions at the second time point:
Suppose we have a pseudometric compact space $(\mathcal{T}, \rho)$ and $n \in \mathbb{N}$ independent realisations

$$\mathbf{x}_1, \ldots, \mathbf{x}_n : \mathcal{T} \to \mathbb{R}^2$$

of a stochastic process $\mathbf{X}$ and write for every $t \in \mathcal{T}$ and $n = n_1 + n_2$:

$$\underbrace{\begin{pmatrix} x_{11}^{(c)}(t) \\ x_{21}^{(c)}(t) \end{pmatrix}, \ldots, \begin{pmatrix} x_{1n_1}^{(c)}(t) \\ x_{2n_1}^{(c)}(t) \end{pmatrix}}_{\mathbf{x}^{(c)}, \text{ of size } n_1.}, \underbrace{\begin{pmatrix} x_{11}^{(m)}(t) \\ --- \end{pmatrix}, \ldots, \begin{pmatrix} x_{1n_2}^{(m)}(t) \\ --- \end{pmatrix}}_{\mathbf{x}^{(m)}, \text{ of size } n_2.}.$$

Hypothesis and Model

- **Same Hypothesis:** functional matched pair hypothesis regarding the pointwise means $\eta_1(t)$ and $\eta_2(t)$, $t \in \mathcal{T}$:

$$\mathcal{H}_0 : \forall\, t \in \mathcal{T} : \eta_1(t) = \eta_2(t), \quad \mathcal{H}_1 : \exists\, t \in \mathcal{T} : \eta_1(t) \neq \eta_2(t). \tag{3}$$

- Missing observations are missing completely at random (MCAR).
- Stochastic process $\mathbf{X}$ are considered as valued in $C^2(\mathcal{T})$ with no specific distribution, but mean function $\boldsymbol{\eta}(t) = (\eta_1(t), \eta_2(t))$, $t \in \mathcal{T}$, and covariance matrix $\boldsymbol{\Gamma}$.
- **Assumptions:** set equivalent to the assumptions for missingness on two arms.
- Similar approach for non-functional data: Amro et al. (2021).

Statistical Methods: Two-sided Missingness in functional Two-sample Testing

Estimators: Pointwise Mean and Covariance I

- Consider two estimators to estimate the mean difference $\eta_1(t) - \eta_2(t)$:

$$\hat{\eta}_1^{(c)} - \hat{\eta}_2^{(c)} \text{ and } \hat{\eta}_1^{(m)} - \hat{\eta}_2^{(c)}.$$

- To jointly handle both, define the vector

$$\hat{\boldsymbol{\mu}}(t) := \left(\hat{\boldsymbol{\eta}}^{(c)}(t)^\top, \hat{\eta}_1^{(m)}(t)\right)^\top = \left(\hat{\eta}_1^{(c)}(t), \hat{\eta}_2^{(c)}(t), \hat{\eta}_1^{(m)}(t)\right)^\top$$

and the contrast matrix

$$\mathbf{A} := \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \in \mathbb{R}^{2\times 3}.$$

$\longrightarrow$ $\mathbf{A}\hat{\boldsymbol{\mu}} = (\hat{\eta}_1^{(c)} - \hat{\eta}_2^{(c)}, \hat{\eta}_1^{(m)} - \hat{\eta}_2^{(c)})$, $\mathcal{H}_0 :$ $\mathbf{A}\boldsymbol{\mu}(t) = \mathbf{0}$, $t \in \mathcal{T}$, $\boldsymbol{\mu}(t) = (\boldsymbol{\eta}(t)^\top, \eta_1(t))^\top$.

Estimators: Pointwise Mean and Covariance II

- Under the Assumptions, it holds

$$\left(\sqrt{n}\mathbf{A}\left(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}\right)(t)\right)_{t\in\mathcal{T}} \xrightarrow{d} \mathbf{Z} \tag{4}$$

in $C^2(\mathcal{T})$, where $\mathbf{Z} \sim \mathrm{GP}_2(\mathbf{0}, \mathrm{Cov}(\mathbf{Z}))$ with $\mathrm{Cov}(\mathbf{Z}(t), \mathbf{Z}(s)) = \mathbf{A}\boldsymbol{\Sigma}(s,t)\mathbf{A}^\top$ and

$$\boldsymbol{\Sigma}(s,t) = \begin{pmatrix} \kappa_1^{-1}\boldsymbol{\Gamma}(s,t) & \mathbf{0}_2 \\ \mathbf{0}_2^\top & \kappa_2^{-1}\gamma_1(s,t) \end{pmatrix}.$$

- Consistent plug-in estimator:

$$\hat{\boldsymbol{\Sigma}}(t,t) = \begin{pmatrix} \frac{n}{n_1}\hat{\boldsymbol{\Gamma}}^{(c)}(t,t) & \mathbf{0}_2 \\ \mathbf{0}_2^\top & \frac{n}{n_2}\hat{\gamma}_1(t,t) \end{pmatrix}. \tag{5}$$

The Test's Construction: Three Pointwise Test Statistics and Globalization

- Construct pointwise quadratic form-type test statistics:

$$T_n(\mathbf{A}, \mathbf{x}, t) = n\,(\mathbf{A}\hat{\boldsymbol{\mu}}(t))^\top \Xi\left(\mathbf{A}, \hat{\boldsymbol{\Sigma}}, t\right) \mathbf{A}\hat{\boldsymbol{\mu}}(t),\ t \in \mathcal{T}.$$

$\longrightarrow$ $\Xi(\boldsymbol{A}, \hat{\boldsymbol{\Sigma}}, t)$ determines the type of test statistic:

- Wald-type statistic (Wald, 1943): $\Xi_{\mathrm{WTS}}(\mathbf{A}, \hat{\boldsymbol{\Sigma}}, t) = (\mathbf{A}\hat{\boldsymbol{\Sigma}}(t,t)\mathbf{A}^\top)^+$.
- ANOVA-type statistic (Brunner et al., 1997): $\Xi_{\mathrm{ATS}}(\mathbf{A}, \hat{\boldsymbol{\Sigma}}, t) = \mathrm{tr}(\mathbf{A}\hat{\boldsymbol{\Sigma}}(t,t)\mathbf{A}^\top)^{-1}$.
- Modified ANOVA-type statistic (Friedrich & Pauly, 2018): $\Xi_{\mathrm{MATS}}(\mathbf{A}, \hat{\boldsymbol{\Sigma}}, t) = (\mathbf{A}\hat{\boldsymbol{\Sigma}}(t,t)\mathbf{A}^\top)_0^+$

$\longrightarrow$ Combine them with integral $\int_{\mathcal{T}} \cdot\, dt$ and supremum $\sup_{t\in\mathcal{T}}\{\cdot\}$.

$\mathbf{B}^+$: Moore-Penrose inverse of a matrix $\mathbf{B}$.

$\mathbf{B}_0$: diagonal matrix having the diagonal elements of $\mathbf{B}$ and zeros elsewhere.

The Test's Construction: Asymptotic Inference

Theorem

Let the Assumptions hold.

1. *Under $\mathcal{H}_0$, we have the following convergence in distribution:*

$$I_n(\mathbf{A}, \mathbf{x}) = \int_{\mathcal{T}} T_n(\mathbf{A}, \mathbf{x}, t)\, dt \xrightarrow{d} \int_{\mathcal{T}} \mathbf{Z}(t)^\top \Xi(\mathbf{A}, \mathbf{\Sigma}, t)\, \mathbf{Z}(t) dt,$$

$$S_n(\mathbf{A}, \mathbf{x}) = \sup_{t \in \mathcal{T}} T_n(\mathbf{A}, \mathbf{x}, t) \xrightarrow{d} \sup_{t \in \mathcal{T}} \left\{ \mathbf{Z}(t)^\top \Xi(\mathbf{A}, \mathbf{\Sigma}, t)\, \mathbf{Z}(t) \right\},$$

   where $\{\mathbf{Z}(t) : t \in \mathcal{T}\}$ is the Gaussian process from Equation (4).

2. *Under $\mathcal{H}_1$, $I_n(\mathbf{A}, \mathbf{x})$ and $S_n(\mathbf{A}, \mathbf{x})$ converge in probability to ∞.*

The Test's Construction: Testing Procedures

$\longrightarrow$ Resampling Methods need to be considered.

- **Parametric Bootstrap Procedure** is theoretically valid by Janssen and Pauls (2003) and Munko et al. (2025).
- Testing Procedure:

$$\varphi_n^{*,I} = \mathbf{1}\left\{I_n(\mathbf{A}, \mathbf{x}) > q_{1-\alpha}^{*,I}(\mathbf{x})\right\}; \quad \varphi_n^{*,S} = \mathbf{1}\left\{S_n(\mathbf{A}, \mathbf{x}) > q_{1-\alpha}^{*,S}(\mathbf{x})\right\}.$$

Simulations

Simulation Set-Up

Consider the data generation model for $t \in [0,1]$, $j \in \{1,\ldots,n\}$ and $i \in \{1,2\}$:

$$x_{ij}(t) = c_i(t) + e_{ij}(t).$$

- Random errors $e_{ij}(t)$ based on standard Brownian Bridges with different correlations ρ.
  - Under the null:
  $$c_1(t) = c_2(t) = (\sin(2\pi t^2))^5 I_{[0,1]}(t).$$
  - Under the alternative:
  $$c_1(t) = (\sin(2\pi t^2))^5 I_{[0,1]}(t),\ c_2(t) = (\sin(2\pi t^2))^3 I_{[0,1]}(t).$$
- Small and moderate sample sizes (span $n = 15$ to $n = 60$) for two-sided missingness and a fixed sample size ($n = 30$) with different numbers of incomplete pairs for one-sided missingness.
- Missing values: randomly inserted by a MCAR mechanism until the number of missing functions is achieved.
- Number of extracted points: $b \in \{25, 50\}$.

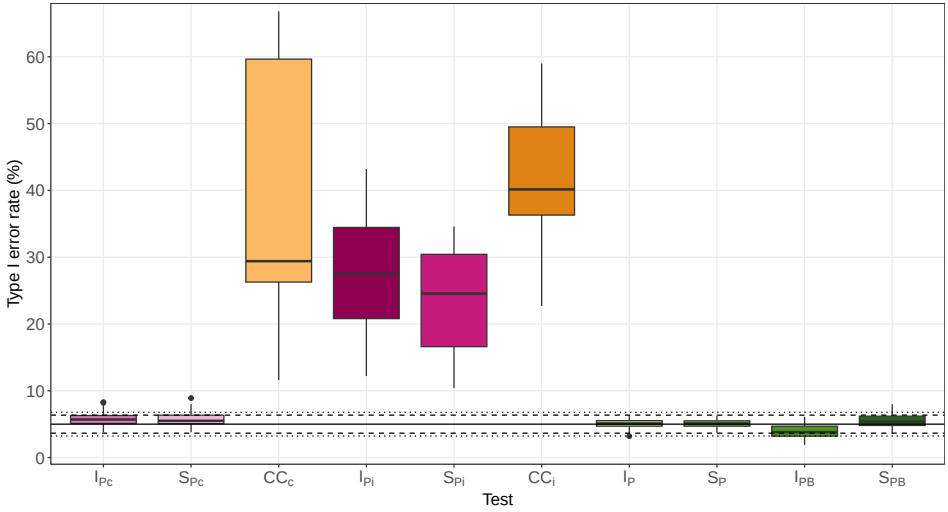
Simulation Two-Sided: Benchmark and New Methods

- I_{Pc} and S_{Pc}: permutation tests based on integral and supremum test statistics, by Smaga (2020) applied to the complete data $\mathbf{x}^{(\mathbf{c})}$.
- CC_c: bootstrap test by Crainiceanu et al. (2012) applied to complete data $\mathbf{x}^{(\mathbf{c})}$.

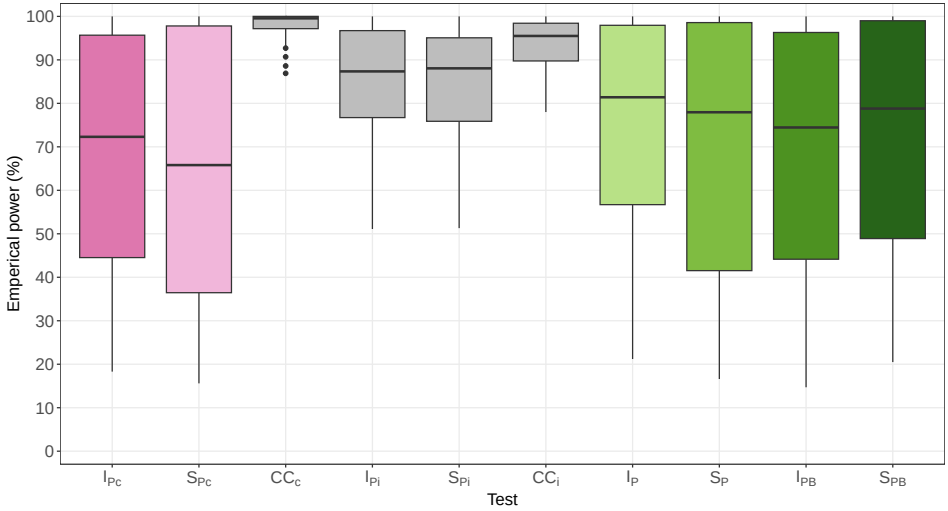
$\longrightarrow$ Do not use the whole data.

- I_{Pi}, S_{Pi}, CC_i: tests above applied to imputed data using the method by Jang et al. (2021).
- I_P and S_P: new permutation tests based on integral and supremum test statistics.
- I_{PB} and S_{PB}: new parametric bootstrap tests based on integral and supremum test statistics.

Simulation Results: Type I Error Brownian Bridges



Simulation Results: Power Brownian Bridges



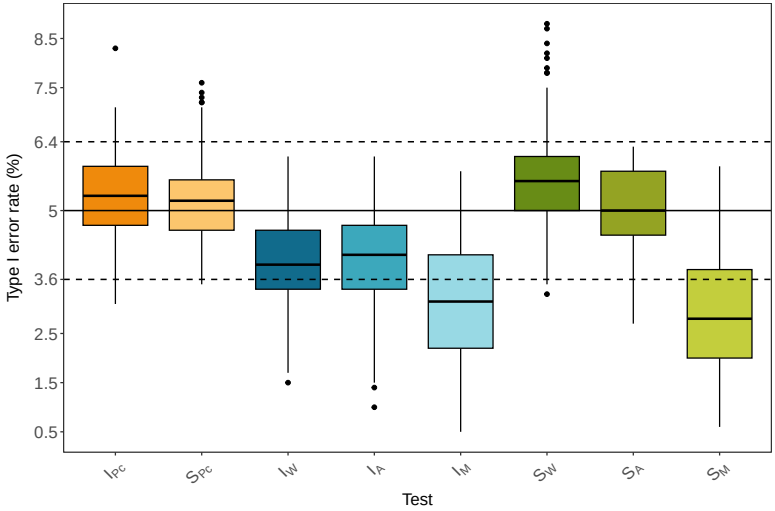
Simulation One-Sided: Benchmark and New Methods

- I_{Pc} and S_{Pc}: permutation tests based on integral and supremum test statistics, by Smaga (2020) applied to the complete data $\mathbf{x}^{(\mathbf{c})}$.

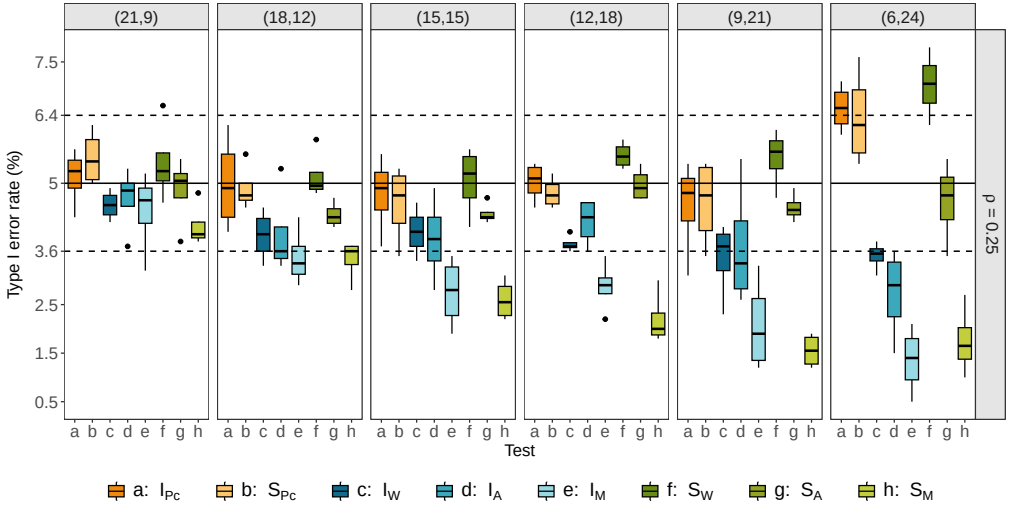
$\longrightarrow$ Do not use the whole data.

- I_W, I_A, and I_M: point-wise WTS, ATS, and MATS combined with integral function.
- S_W, S_A, and S_M: point-wise WTS, ATS, and MATS combined with supremum function.

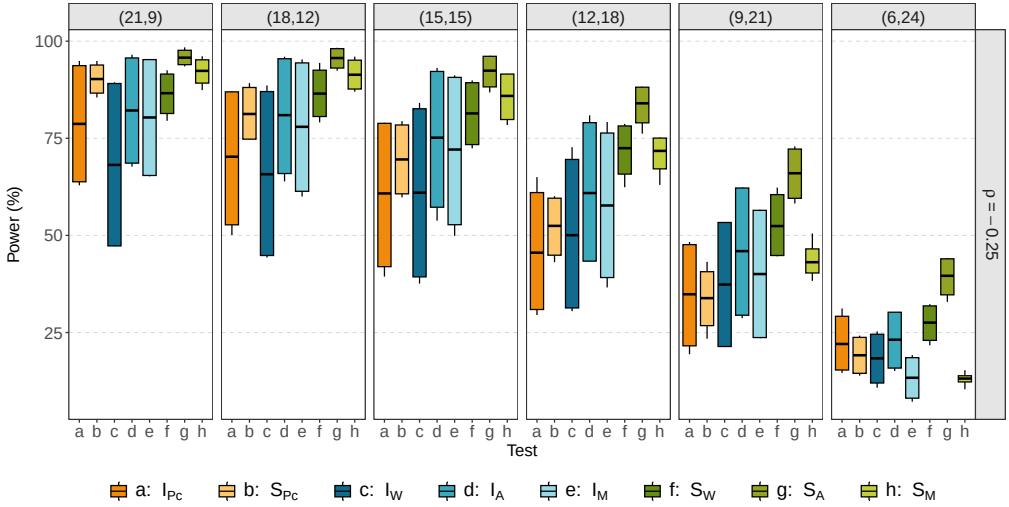
Simulation Results: Type I Error



Simulation Results: Type I Error Divided by Sample Size



Simulation Results: Power Divided by Sample Size



Summary

Summary

- Powerful methods to consider missingness in functional data.
- Adaption of the permutation approach in Amro and Pauly (2017) is not theoretically valid.

$\longrightarrow$ We have proven the permutation method in Smaga (2020).

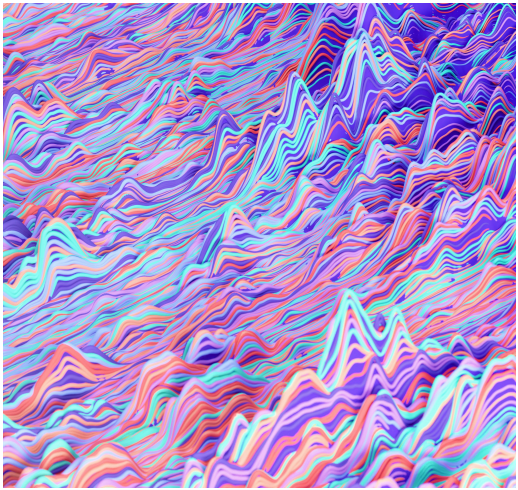
- Parametric bootstrap approach is theoretically valid for both methods.
  - Missingness in both arms: integral and supremum statistics control level α, power depends highly on the data.
  - Missingness in one arm: supremum statistics with ATS (S_A) controls level alpha best, power depends highly on the amount of missing functions.

Thank you for your attention!

arXiv-Preprint

Supplement on TUDOdata

marlene-baumeister.de



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Assumptions

1. The subject-effect functions $\mathbf{v}_j(t) = \mathbf{x}_j(t) - \boldsymbol{\eta}(t)$, $j \in \{1, \ldots, n\}$, $t \in \mathcal{T}$, are i.i.d. realisations of a stochastic process $\{\mathbf{V}(t) : t \in \mathcal{T}\}$, $\mathbf{V} : \Omega \to C^2(\mathcal{T})$ with zero mean function and covariance function $\boldsymbol{\Gamma}$.
2. There exist $t^* \in \mathcal{T}$ and $J \leq 2$ such that $\mathrm{E}[\|\mathbf{v}_1(t^*)\|_2^J] < \infty$, where here and throughout $\|\cdot\|_2$ denotes the Euclidean norm of a vector of numbers.
3. There exists a $C \geq 0$ for all $s, t \in \mathcal{T}$ such that $\mathrm{E}[\|\mathbf{v}_1(t) - \mathbf{v}_1(s)\|_2^J] \leq C \cdot \rho(s,t)^J$, and there is a $\nu > 0$ such that

$$\int_0^\nu \mathrm{D}(\epsilon, \rho)^{2/J} d\epsilon < \infty. \tag{6}$$

Here, and throughout, J is chosen as in (2) and $\mathrm{D}(\epsilon, \rho)$ denotes the packing number, see van der Vaart and Wellner (2023, Def. 2.2.3).

Covariances of the Limiting Processes of the Paired and Unpaired Test Statistics

$\mathrm{Var}(Z_i) = 1$, $i \in \{1, 2\}$, for all $s, t \in \mathcal{T}$ and:

- $Z_1 \sim \mathrm{GP}(0, \mathrm{Cov}(Z_1))$ with

$$\mathrm{Cov}(Z_1(t), Z_1(s)) = \frac{\gamma_{1-2}(s,t)}{\sqrt{\gamma_{1-2}(s)\gamma_{1-2}(t)}}.$$

- $Z_2 \sim \mathrm{GP}(0, \mathrm{Cov}(Z_2))$ with

$$\mathrm{Cov}(Z_2(t), Z_2(s)) = \frac{\frac{1}{\kappa_2}\gamma_1(s,t) + \frac{1}{\kappa_3}\gamma_2(s,t)}{\sqrt{\frac{1}{\kappa_2}\gamma_1(t,t) + \frac{1}{\kappa_3}\gamma_2(t,t)}\sqrt{\frac{1}{\kappa_2}\gamma_1(s,s) + \frac{1}{\kappa_3}\gamma_2(s,s)}}.$$